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Plurisubharmonicity for the solution of the Fefferman equation and applications

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Abstract In this paper, the author introduces a concept of the super-pseudoconvex domain. He proves that the solution of the Fefferman equation on a smoothly bounded strictly pseudoconvex domain D in $\mathbb{C}^n$ is plurisubharmonic in D if and only if D is super-pseudoconvex. As an application, when D is super-pseudoconvex, he gives the sharp lower bound for the bottom of the spectrum of the Laplace-Beltrami operators by using the result of Li and Wang (Int. Math. Res. Not. 4351–4371, 2012).

Keywords Kähler–Einstein · Monge–Ampère · Plurisubharmonic · Bottom of spectrum

1 Introduction

Let D be a smoothly bounded pseudoconvex domain D in $\mathbb{C}^n$. Let $u \in C^2(D)$ be a real-valued function and let $H(u)$ denote the $n \times n$ complex Hessian matrix of u. We say that u is strictly plurisubharmonic in D if $H(u)$ is positive definite on D. When u is strictly plurisubharmonic in D, u induces a Kähler metric

$$g = g[u] = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz^i \otimes d\bar{z}^j. \tag{1.1}$$

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We say that the metric g is also Einstein if its Ricci curvature

$$R_{k\bar{\ell}} = -\frac{\partial^2 \log \det[g_{i\bar{j}}]}{\partial z_k \partial \bar{z}_\ell} \tag{1.2}$$

satisfies the equation: $R_{k\bar{\ell}} = c g_{k\bar{\ell}}$ for some constant c.

When $c < 0$, after a normalization, we may assume $c = -(n+1)$. It was proved by Cheng and Yau [5] that the following Monge–Ampère equation:

$$\begin{cases} \det H(u) = e^{(n+1)u}, & z \in D \\ u = +\infty, & z \in \partial D \end{cases} \tag{1.3}$$

has a unique strictly plurisubharmonic solution $u \in C^\infty(D)$. Moreover, the Kähler metric

$$g[u] = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j \tag{1.4}$$

induced by u is a complete Kähler–Einstein metric on D.

When D is also strictly pseudoconvex, the existence and uniqueness problem was studied by Fefferman [6] earlier. He considered the following Fefferman equation

$$\begin{cases} \det J(\rho) = 1, & z \in D, \\ \rho = 0, & z \in \partial D, \end{cases} \tag{1.5}$$

where

$$J(\rho) = -\det \begin{bmatrix} \rho & \bar{\partial}\rho \\ (\bar{\partial}\rho)^* & H(\rho) \end{bmatrix}, \quad \bar{\partial}\rho = \left(\frac{\partial \rho}{\partial \bar{z}_1}, \dots, \frac{\partial \rho}{\partial \bar{z}_n} \right) \text{ and}$$
$$(\bar{\partial}\rho)^* = \left(\frac{\partial \rho}{\partial z_1}, \dots, \frac{\partial \rho}{\partial z_n} \right)^t. \tag{1.6}$$

Fefferman searched for a solution $\rho < 0$ on D such that $u = -\log(-\rho)$ is strictly plurisubharmonic in D. He proved the uniqueness and gave a formal or approximation solution for (1.5).

If the relation between ρ and u is given by

$$\rho(z) = -e^{-u(z)}, \qquad z \in D, \tag{1.7}$$

then (1.3) is the same as (1.5). Moreover, one can prove (see [14] and references therein) that

$$\det H(u) = J(\rho) e^{(n+1)u}. \tag{1.8}$$

When D is smoothly bounded strictly pseudoconvex, it was proved by Cheng and Yau [5] that $\rho \in C^{n+3/2}(\overline{D})$. In fact, $\rho \in C^{n+2-\epsilon}(\overline{D})$ for any small $\epsilon > 0$. This follows from an asymptotic expansion formula for ρ obtained by Lee and Melrose [10]:

$$\rho(z) = r(z)\Big(a_0(z) + \sum_{j=1}^{\infty} a_j(r^{n+1}\log(-r))^j\Big), \tag{1.9}$$

where $r \in C^\infty(\overline{D})$ is any defining function for D and $a_j \in C^\infty(\overline{D})$ and $a_0(z) > 0$ on ∂D.

When D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r, one can view $(\partial D, \theta)$ as a pseudo-Hermitian CR manifold with the contact/pseudo Hermitian form

$$\theta = \frac{1}{2i}(\partial r - \overline{\partial} r). \tag{1.10}$$

An interesting and useful question is: How to find a defining function r such that $(\partial D, \theta)$ has positive the Webster-Tanaka pseudo Ricci curvature or pseudo scalar curvature? Under the assumption $u = -\log(-r)$ is strictly plurisubharmonic near and on ∂D, the following formula for the pseudo-Ricci curvature was discovered by Li and Luk [18]:

$$Ric_z(w, \overline{v}) = -\sum_{k,\ell=1}^{n} \frac{\partial^2 \log J(r)(z)}{\partial z_k \partial \overline{z}_\ell} w_k \overline{v}_\ell + n\frac{\det H(r)}{J(r)} \sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_k \partial \overline{z}_\ell} w_k \overline{v}_\ell \tag{1.11}$$

for $w, v \in H_z = \{v = (v_1, \ldots, v_n) \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} v_j = 0\}$.

When $g[u]$ is asymptotic Einstein (i.e. $J(r) = 1 + O(r^2)$), one has that

$$Ric_z(w, \overline{v}) = n\frac{\det H(r)}{J(r)} \sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_k \partial \overline{z}_\ell} w_k \overline{v}_\ell \tag{1.12}$$

for $w, v \in H_z = \{v = (v_1, \ldots, v_n) \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r(z)}{\partial z_j} v_j = 0\}$. In this case, the Webster-Tanaka pseudo-Hermitian metric is a pseudo Einstein metric. Moreover, the pseudo Ricci curvature is positive on ∂D if and only if $\det H(r) > 0$ on ∂D.

Many researches [14,15,19,20] indicate that the following problem is very interesting and very important.

Problem 1 Assume that D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let ρ be the solution of the Fefferman equation (1.5) such that $u = -\log(-\rho)$ is strictly plurisubharmonic in D. For what extra condition on D, one has that ρ is strictly plurisubharmonic in $\overline{D}$.

It is well known that $\rho(z) = |z|^2 - 1$ is strictly plurisubharmonic when $D = B_n$, the unit ball in $\mathbb{C}^n$. It was proved by the Li [14] that ρ is strictly plurisubharmonic when D is a bounded domain in $\mathbb{C}^n$ whose boundary is a real ellipsoid. In particular, when $n = 2$ case, this result was also proved by Chanillo, Chiu and Yang [2] later.

One of the main purposes of this paper is to give a characterization for domains D in $\mathbb{C}^n$ where the answer of Problem 1 is affirmatively true. We first introduce the following definition.

Definition 1.1 Let D be a smoothly bounded domain in $\mathbb{C}^n$. We say that D is strictly super-pseudoconvex (super-superconvex) if there is a strictly plurisubharmonic defining function $r \in C^4(\overline{D})$ such that $\mathcal{L}_2[r] > 0$ ($\mathcal{L}_2[r] \geq 0$) on ∂D, respectively. Here

$$\mathcal{L}_2[r] =: 1 + \frac{|\partial r|_r^2}{n(n+1)} \tilde{\Delta} \log J(r) - \frac{2\mathrm{Re}\, R \log J(r)}{n+1} - |\partial r|_r^2 |\tilde{\nabla} \log J(r)|^2, \quad (1.13)$$

with

$$\tilde{\Delta} = \sum_{i,j=1}^n a^{i\bar{j}}[r] \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \quad R = \sum_{j=1}^n r^j \frac{\partial}{\partial z_j}, \quad |\tilde{\nabla} f|^2 = \sum_{i,j=1}^n a^{i\bar{j}}[r] \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} \quad (1.14)$$

and

$$r^i = \sum_{j=1}^n r^{i\bar{j}} r_{\bar{j}}, \quad \left[r^{i\bar{j}}\right]^t = H(r)^{-1}, \quad a^{i\bar{j}}[r] =: r^{i\bar{j}} - \frac{r^i r^{\bar{j}}}{-r + |\partial r|_r^2}, \quad 1 \leq i, j \leq n. \quad (1.15)$$

Another motivation of this paper is to apply the result (the solution of Problem 1) to estimate the lower bound of the bottom of the spectrum of Laplace-Beltrami operator $\Delta_{g[u]}$.

Definition 1.2 Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $r \in C^\infty(\overline{D})$ be a defining function for D such that $u = -\log(-r)$ is strictly plurisubharmonic. We say that the Kähler metric $g[u]$ induced by u is **super asymptotic Einstein** if

(i) the Ricci curvature $R_{i\bar{j}} \geq -(n+1) g_{i\bar{j}}$ on D; and
(ii) $J(r) = 1 + O(r^2)$.

Let (M^n, g) be a Kähler manifold with the Kähler metric g. Let Δ_g be the Laplcae-Beltrami operator associated to g. Let λ_1 denote the bottom of the spectrum of Δ_g. Then the problem of estimating the upper bound and lower bound for λ_1 have studied by many authors, including Cheng [4], Lee [9], Li and Wang [12,13], Munteanu [22], Li and Tran [19] and Li and Wang [20], Wang [24], etc... When the Ricci curvature is super Einstein: $R_{i\bar{j}} \geq -(n+1) g_{i\bar{j}}$, Munteanu [22] proves that $\lambda_1 \leq n^2$. For the lower bound estimate of λ_1, Li and Tran [19] and Li and Wang [20] consider a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with defining function $r \in C^4(\overline{D})$ such that $u =: -\log(-r)$ is strictly plurisubharmonic in D. When r is plurisubharmonic in D, Li and Tran [19] prove that $\lambda_1 = n^2$. When $g[u]$ is *super asymptotic Einstein* and $\det H(r) \geq 0$ on ∂D, Li and Wang [20] prove $\lambda_1 = n^2$. We will show that $\det H(r) \geq 0$ on ∂D when D is super-pseudoconvex.

The first result of the paper is the following theorems.

Theorem 1.3 *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\tilde{\rho} \in C^4(\overline{D})$ be a defining function for D such that $\tilde{u} = -\log(-\tilde{\rho})$ is strictly plurisubharmonic. If the Kähler metric $g[\tilde{u}]$ induced by $\tilde{u}$ is the super asymptotic Einstein, then the following two statements hold:*

(i) $\tilde{\rho}$ is strictly plurisubharmonic on $\overline{D}$ if and only if D is strictly super-pseudoconvex. In particular if $\tilde{\rho} = \rho(z)$ is the solution of (1.5) then ρ is strictly plurisubharmonic in $\overline{D}$ when D is strictly super-pseudoconvex;
(ii) If D is also super-pseudoconvex then $\lambda_1(\Delta_{g[\tilde{u}]}) = n^2$, where $\Delta_g = -4\sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$.

It is interesting to bridge the relation between convex and super-pseudoconvex. The second result of the paper is:

Theorem 1.4 *Let D be a smoothly bounded domain in $\mathbb{C}^n$. Then*

(i) When $n = 1$, D is strictly super-pseudoconvex (super-pseudoconvex) if and only if D is strictly convex (convex);
(ii) When $n > 1$, if D is convex and if there is a strictly plurisubharmonic defining function $r \in C^4(\overline{D})$ such that

$$n-1+\frac{|\partial r|^2}{n} a^{k\bar{\ell}}[r]\Big[\tilde{\Delta} r_{k\bar{\ell}} - a^{i\bar{q}}[r] r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} - (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}})\Big] - 2\mathrm{Re}\, r^k \tilde{\Delta} r_k > 0, \tag{1.16}$$

then D is strictly super-pseudoconvex;
(iii) The convexity does not imply super-pseudoconvexity and the super-pseudoconvexity does not imply the convexity either.

The paper is organized as follows: Sect. 2, we give an approximation formula. Theorem 1.3 will be proved in Sect. 3; Part (i) and Part (ii) of Theorem 1.4 will be proved in Sect. 4. Finally, in Sect. 5, we provide two domains in $\mathbb{C}^2$; one is strictly convex but not super-pseudoconvex and the other is super-pseudoconvex but not convex. These prove Part (iii) of Theorem 1.4.

2 An approximation formula

Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Let $r \in C^2(\overline{D})$ be a real-valued, negative defining function for D. Then the Fefferman operator [5,6] acting on

r is defined by

$$J(r) = -\det \begin{bmatrix} r & \overline{\partial} r \\ (\overline{\partial} r)^* & H(r) \end{bmatrix}, \tag{2.1}$$

where $\overline{\partial} r = (\frac{\partial r}{\partial \overline{z}_1}, \ldots, \frac{\partial r}{\partial \overline{z}_n}) = (r_{\overline{1}}, \ldots, r_{\overline{n}})$ is a row vector in $\mathbb{C}^n$ and $(\overline{\partial} r)^*$ is its adjoint vector, which is column vector in $\mathbb{C}^n$ and $H(r) = [\frac{\partial^2 r}{\partial z_i \partial \overline{z}_j}]$ is the $n \times n$ complex Hessian matrix of r.

If $H(r) = [r_{i\overline{j}}]$ is invertible, in particular it is positive definite, then we use the notation $[r^{i\overline{j}}]^t =: H(r)^{-1}$ and

$$|\partial r|_r^2 = \sum_{i,j=1}^n r^{i\overline{j}} r_i r_{\overline{j}}. \tag{2.2}$$

It is easy to verify that

$$J(r) = \det H(r)(-r + |\partial r|_r^2). \tag{2.3}$$

In fact, since

$$\begin{aligned} J(r) &= (-r) \det \left[H(r) - \frac{(\overline{\partial} r)^* (\overline{\partial} r)}{r} \right] \\ &= (-r) \det H(r) \left(1 - \frac{|\partial r|_r^2}{r} \right) \\ &= \det H(r) \left(-r + |\partial r|_r^2 \right). \end{aligned} \tag{2.4}$$

Remark 1 When $H(r)$ is not positive definite on ∂D, we can replace r by

$$r[a] =: r(z) + \frac{a}{2} r^2. \tag{2.5}$$

Then $r[a]$ is positive definite with a large a and

$$J(r) = \frac{1}{(1 + ar)^n} \det H(r[a])(-r + (1 + 2a\, r)|\partial r|_{r[a]}). \tag{2.6}$$

From now on, we will always assume that $r(z) \in C^\infty(\overline{D})$ is a negative defining function for D such that

$$\ell(r) = -\log(-r) \tag{2.7}$$

is strictly plurisubharmonic in D. It is known from [5,14–16] that the following identity holds:

$$\det H(\ell(r)) = J(r)e^{(n+1)\ell(r)}. \tag{2.8}$$

This implies that

(i) $u =: \ell(r)$ is strictly plurisubharmonic on D if and only if $J(r) > 0$ on D;
(ii) $J(r) = 1$ if and only if $\det H(u) = e^{(n+1)u}$ with $u =: \ell(r)$.

Fefferman [6] gave a formula to approximate the potential function ρ [for Eq. (1.5)]. He proved that $J(r\, J(r)^{-1/(n+1)}) = 1 + O(r)$ near ∂D. Higher order approximation can be iterated through the previous steps. Based on the Fefferman's idea, the iteration formula of the approximation was given in more detail by Graham in [7]. The author [14] gave another modification. For convenience of readers and further argument for the current paper, we will state and prove a second order approximation formula here.

Theorem 2.1 *Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. Let $r(z)$ be a smooth negative defining function for D such that $\ell(r)$ is strictly plurisubharmonic in D. Let*

$$\rho_1(z) = r(z)J(r)^{-1/(n+1)}e^{-B(z)} \tag{2.9}$$

with

$$B(z) = B[r](z) = \frac{tr(H(\ell(r))^{-1}H(\log J(r))}{2n(n+1)}. \tag{2.10}$$

Then

$$J(\rho_1)(z) = 1 + O(r^2). \tag{2.11}$$

Moreover, if $J(r) = 1 + O(r^2)$ then $\rho_1 = r + O(r^3)$ and $J(\rho_1) = 1 + O(r^3)$.

Proof Since

$$H(\ell(r)) = \frac{1}{(-r)(1+ar)}\left[H(r_a) + \frac{1+2a\,r}{(-r)}(\bar\partial r)^*(\bar\partial r)\right] \tag{2.12}$$

by choosing $a \geq 0$ so that $r[a]$ is strictly plurisubharmonic. Therefore, we can write

$$B(z) = (-r)B_0(z), \tag{2.13}$$

with $B_0 \in C^\infty(\overline{D})$. Since

$$\begin{aligned} H(B) = (-r)H(B_0) - B_0\left(H(r) + \frac{(\bar\partial r)^*\bar\partial r}{-r}\right) + B_0\frac{(\bar\partial r)^*\bar\partial r}{-r} - (\bar\partial r)^*(\bar\partial B_0) \\ -(\bar\partial B)^*(\bar\partial r). \end{aligned} \tag{2.14}$$

By complex rotation, one may assume that $\frac{\partial r}{\partial z_j}(z_0) = 0$ for $1 \leq j \leq n-1$ and $H(r)(z_0)$ is diagonal, it is easy to verify that

$$\mathrm{tr}(H(\ell(r)^{-1}H(B)) = -nB(z) + (-r)B_0 + O(r^2) = -(n-1)B + O(r^2). \tag{2.15}$$

Since

$$\begin{aligned}
J(\rho_1)(z)e^{(n+1)\ell(\rho_1)} &= \det H(\ell(\rho_1)) \\
&= \det\left(H(\ell(r)) + \frac{1}{n+1}H(\log J) + H(B)\right) \\
&= \det H(\ell(r)) \det\left(I_n + H(\ell(r))^{-1}\left[\frac{1}{n+1}H(\log J) + H(B)\right]\right) \\
&= J(r)e^{(n+1)\ell(r)} \det\left(I_n + H(\ell(r))^{-1}\left[\frac{1}{n+1}H(\log J) + H(B)\right]\right)
\end{aligned}$$

Notice that $\exp((n+1)\ell(\rho_1)) = \exp((n+1)B)J(r)\exp((n+1)\ell(r))$, we have

$$\begin{aligned}
J(\rho_1)(z) &= e^{-(n+1)B} \det\left(I_n + H(\ell(r))^{-1}\left[\frac{1}{n+1}H(\log J) + H(B)\right]\right) \\
&= e^{-(n+1)B}\left[1 + \mathrm{tr}\left[H(\ell(r))^{-1}\left[\frac{1}{n+1}H(\log J) + H(B)\right]\right]\right] + O(r^2) \\
&= e^{-(n+1)B}[1 + 2nB + \mathrm{tr}(H(\ell(r))^{-1}H(B)] + O(r^2) \\
&= e^{-(n+1)B}[1 + 2nB - (n-1)B + O(r^2)] + O(r^2) \\
&= 1 + \frac{(n+1)^2}{2}B^2 + O(r^2) \\
&= 1 + O(r^2).
\end{aligned}$$

When $J(r) = 1 + Ar^2$ with A is smooth on $\overline{D}$, it is easy to prove $B = B_1 r^2$ with B_1 smooth in $\overline{D}$ near ∂D. It is also easy to verify that $\rho_1[r] = r + O(r^3)$ and $J(\rho_1[r]) = 1 + O(r^3)$. This proves Theorem 2.1. □

Proposition 2.2 *Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let u be the plurisubharmonic solution of (1.3) and $\rho(z) = -e^{-u}$. Then for any smooth defining function r of D with $\ell(r)$ being strictly plurisubharmonic in D, we have*

$$\begin{aligned}
\det H(\rho) = J(r)^{\frac{-n}{n+1}} \det\Bigg(H(r) - &\frac{[\partial_i r\partial_{\bar{j}} \log J + \partial_i \log J(r)\, \partial_{\bar{j}} r]}{n+1} \\
&-[\partial_i r\partial_{\bar{j}} B(z) + \partial_i B\partial_{\bar{j}} r]\Bigg)
\end{aligned} \tag{2.16}$$

on ∂D, where $B(z) = B[r](z)$ is given by (2.10).

Proof Let

$$\rho_1(z) = \rho_1[r] =: r(z)J(r)^{-1/(n+1)}e^{-B}. \tag{2.17}$$

Theorem 2.1 implies that $\rho(z) = \rho_1(z) + O(r(z)^3)$. A simple calculation shows that

$$\det H(\rho) = \det H(\rho_1), \quad z \in \partial D. \tag{2.18}$$

By (2.13) ($B = (-r)B_0$), one can easily see that

$$\rho_1(z) = r(z)J(r)^{-1/(n+1)} - r(z)J(r)^{-1/(n+1)}B(z) + O(r(z)^3) \tag{2.19}$$

and

$$\det H(\rho_1) = \det H\Big(r(z)J(r)^{-1/(n+1)} - r(z)J(r)^{-1/(n+1)}B(z)\Big), \quad z \in \partial D. \tag{2.20}$$

For any $z \in \partial D$, by (2.20), one has

$$\begin{aligned}
\det H(\rho_1)(z) &= \det\Big(H(rJ(r)^{-1/(n+1)}) - J(r)^{-1/(n+1)}[\partial_i r\partial_{\bar{j}}B + \partial_i B\partial_{\bar{j}}r]\Big) \\
&= \det\Big(J(r)^{\frac{-1}{(n+1)}}H(r) - \frac{J^{\frac{-(n+2)}{(n+1)}}}{n+1}[\partial_i r\partial_{\bar{j}}J + \partial_i J(r)\partial_{\bar{j}}r] \\
&\quad - J(r)^{\frac{-1}{(n+1)}}[\partial_i r\partial_{\bar{j}}B + \partial_i B\partial_{\bar{j}}r]\Big) \\
&= J(r)^{\frac{-n}{n+1}}\det\Big(H(r) - \frac{1}{n+1}[\partial_i r\partial_{\bar{j}}\log J + \partial_i \log J(r)\,\partial_{\bar{j}}r] \\
&\quad - [\partial_i r\partial_{\bar{j}}B + \partial_i B\partial_{\bar{j}}r]\Big).
\end{aligned} \tag{2.21}$$

This proves Proposition 2.2. □

Let u^{D_j} be the potential functions for the Kähler–Einstein metric of D_j and let

$$\rho^{D_j}(z) = -e^{-u^{D_j}(z)}, \quad j = 1, 2. \tag{2.22}$$

Proposition 2.3 *Let $\phi : D_1 \to D_2$ be a smooth biholomorphic mapping. Then*

$$\rho^{D_1}(z) = \rho^{D_2}(\phi(z))|\det\phi'(z)|^{-2/(n+1)} \tag{2.23}$$

In particular, if $\det\phi'(z)$ *is constant* c *then*

$$\det H(\rho^{D_1})(z) = |c|^{2/(n+1)}\det H(\rho^{D_2})(\phi(z)). \tag{2.24}$$

Proof Since $\phi : D_1 \to D_2$ is biholomorphic, one has that if u^{D_j} is the unique plurisubharmonic solutions for the Monge–Ampère equation:

$$\begin{cases} \det H(u) = e^{(n+1)u}, & z \in D_j \\ u = \infty, & z \in \partial D_j \end{cases} \tag{2.25}$$

Then

$$u^{D_1}(z) = u^{D_2}(\phi(z)) + \frac{1}{n+1}\log|\det\phi'(z)|^2, \quad z \in D_1 \tag{2.26}$$

and

$$\rho^{D_1}(z) = \rho^{D_2}(\phi(z))|\det\phi'(z)|^{-2/(n+1)}. \tag{2.27}$$

In particular, when $\det\phi'(z) = c$, one has

$$\det H(\rho^{D_1})(z) = |c|^{-2n/(n+1)}\det H(\rho^{D_2})(\phi(z))|c|^2 = |c|^{2/(n+1)}\det H(\rho^{D_2})(\phi(z))$$

and the proof of Proposition 2.3 is complete. □

We also need the following holomorphic change of variables formula.

Lemma 2.4 *For $z_0 \in \partial D$, if $z = \phi(w) : B(0,\delta_0) \to B(z_0, 1)$ be a one-to-one holomorphic map with $\phi(0) = z_0$ and $r(z) = \tilde r(w)$, then*

$$\rho_1(\phi(w)) = |\det\phi'(w)|^{2/(n+1)}\frac{\tilde r(w)}{J(\tilde r(w))^{1/(n+1)}}e^{-B(\tilde r(w))}. \tag{2.28}$$

Moreover, if $|\det\phi'(z)|^2$ is a constant on $B(0,\delta_0)$ for some $\delta_0 > 0$

$$\det H(\rho_1)(z_0)|\det\phi'(0)|^{\frac{2}{n+1}} = \det H\Big(\frac{\tilde r}{J(\tilde r)^{1/(n+1)}}e^{-B(\tilde r)}\Big)(0). \tag{2.29}$$

Proof Since $|\det\phi'(z)|^2$ is constant, by the definitions for $B[r]$ and $J(r)$ from Theorem 2.1, one can easily prove (2.27) and (2.29), and the proposition is proved. □

3 Proof of Theorem 1.3

Let D be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $r \in C^\infty(\overline{D})$ be any strictly plurisubharmonic defining function for D. Let

$$\rho_1(z) = r(z)J(r)^{-1/(n+1)}\exp(-B(z)) \tag{3.1}$$

where

$$B(z) = \frac{\mathrm{tr}(H(\ell(r))^{-1}H(\log J(r))}{2n(n+1)}, \tag{3.2}$$

According to Theorem 2.1, one has

$$J(\rho_1) = 1 + O(r(z)^2). \tag{3.3}$$

Let $\rho = \rho^D$ be the solution of (1.5) such that $\ell(\rho)$ is strictly plurisubharmonic in D. Then

$$\det H(\rho)(z) = \det H(\rho_1)(z) \quad \text{on } \partial D. \tag{3.4}$$

By Proposition 2.2 and

$$\begin{aligned} B(z) &= \frac{(-r)}{2n(n+1)} \mathrm{tr}\left[\left(H(r) + \frac{r_i r_{\overline{j}}}{-r}\right)^{-1} H(\log J(r))\right](z) \\ &= \frac{(-r)}{2n(n+1)} \sum_{j,k=1}^{n} \left(r^{i\overline{j}} - \frac{r^i r^{\overline{j}}}{-r + |\partial r|_r^2}\right) \frac{\partial^2 \log J(r)}{\partial z_i \partial \overline{z}_j} = -B^0(z) r, \end{aligned} \tag{3.5}$$

where

$$B^0(z) = \frac{1}{2n(n+1)} \sum_{j,k=1}^{n} a^{i\overline{j}}[r] \frac{\partial^2 \log J(r)}{\partial z_i \partial \overline{z}_j} = \frac{1}{2n(n+1)} \tilde{\Delta}_r \log J(r). \tag{3.6}$$

Thus for $z_0 \in \partial D$, one has

$$\partial_j B(z_0) = -B^0(z_0) \partial_j r(z_0), \quad \partial_{\overline{j}} B(z_0) = -B^0(z_0) \partial_{\overline{j}} r(z_0), \quad \text{for } 1 \leq j \leq n. \tag{3.7}$$

Let

$$R = \sum_{j=1}^{n} r^j \frac{\partial}{\partial z_j}, \quad \overline{R} = \sum_{j=1}^{n} r^{\overline{j}} \frac{\partial}{\partial \overline{z}_j}, \quad r^i = r^{i\overline{j}} r_{\overline{j}}, \quad r^{\overline{j}} = r^{i\overline{j}} r_i. \tag{3.8}$$

and

$$\left|\tilde{\nabla}_r f\right|^2 =: \sum_{i,j=1}^{n} \left(r^{i\overline{j}} - \frac{r^i r^{\overline{j}}}{-r + |\partial r|_r^2}\right) \partial_i f \partial_{\overline{j}} f = \sum_{i,j=1}^{n} r^{i\overline{j}} \partial_i f \partial_{\overline{j}} f - \frac{|Rf|^2}{-r + |\partial r|_r^2}. \tag{3.9}$$

Then it is easy to see that

$$|\tilde{\nabla}_r r|^2 = 0 \quad \text{on } \partial D. \tag{3.10}$$

Therefore, by (2.21) and Lemma 3.1 in [14], at $z = z_0 \in \partial D$, one has

$$\begin{aligned} &\det H(\rho)(z^0)\, J(r)^{n/(n+1)}(z^0) \\ &= \det H(r) \left(\left| 1 - r^{i\overline{j}} \left(\partial_i r \left(\frac{\partial_{\overline{j}} \log J(r)}{n+1} - B^0 \partial_{\overline{j}} r \right) \right) \right|^2 \right) \end{aligned}$$

$$
\begin{aligned}
&-|\partial r|_r^2 \sum_{i,j=1}^n r^{i\bar{j}} \left(\frac{\partial_i \log J(r)}{n+1} - B^0 \partial_i r\right)\left(\frac{\partial_{\bar{j}} \log J(r)}{n+1} - B^0 \partial_{\bar{j}} r\right)\Big)\\
&= \det H(r)\Big(\Big|1 - \frac{\overline{R} \log J(r)}{n+1} + B^0 |\partial r|_r^2\Big|^2\\
&-|\partial r|_r^2 \sum_{i,j=1}^n r^{i\bar{j}} \frac{\partial_i \log J(r) \partial_{\bar{j}} \log J(r)}{(n+1)^2} + |\partial r|_r^2 2\mathrm{Re}\, B^0 \frac{\overline{R} \log J(r)}{n+1} - |\partial r|_r^4 |B^0|^2\Big)\\
&= \det H(r)\Big(1 + 2B^0 |\partial r|^2 - 2\mathrm{Re}\, \frac{\overline{R} \log J(r)}{n+1} - \frac{|\partial r|_r^2}{(n+1)^2} |\tilde{\nabla}_r \log J(r)|^2\Big)\\
&= \det H(r)\Big(1 + \frac{|\partial r|^2}{n(n+1)} \tilde{\Delta} \log J(r) - 2\mathrm{Re}\, \frac{\overline{R} \log J(r)}{n+1} - \frac{|\partial r|_r^2}{(n+1)^2} |\tilde{\nabla}_r \log J(r)|^2\Big)\\
&> 0 \qquad (3.11)
\end{aligned}
$$

since D is strictly super-pseudoconvex, there is a strictly plurisubharmonic function $r \in C^4(\overline{D})$ such that the above inequality holds on ∂D. If $\tilde{\rho}$ is smooth defining function for D such that the Kähler metric induced by $\tilde{u} = -\log(-\tilde{\rho})$ is super asymptotic Enistein, then $\det H(\tilde{\rho}) = \det H(\rho) > 0$ on ∂D by (3.11). By Lemma 2 in [20], one has that $\det H(\tilde{\rho})$ attains its minimum over $\overline{D}$ at some ponit in ∂D. Therefore, $\det H(\tilde{\rho}) > 0$ on $\overline{D}$ and the proof of Part (i) of Theorem 1.3 is complete. Part (ii) of Theorem 1.3 is a corollary of Part (i) and the result in [19] and [20]. Therefore, the proof of Theorem 1.3 is complete. □

4 Super-pseudoconvex domains

In this section, we will study the relation between super-pseudoconvex domains and convex domains when $n = 1$. We will also study and simplify some quantities in the definition of the super-pseudoconvex domain in $\mathbb{C}^n$. Since

$$
\begin{aligned}
\log J(r) &= \log \det H(r) + \log(-r + |\partial r|_r^2), \qquad (4.1)\\
\frac{\partial(-r + |\partial r|_r^2)}{\partial z_k} &= -r_k + \partial_k(r^{i\bar{j}}) r_i r_{\bar{j}} + r^{i\bar{j}} r_{ik} r_{\bar{j}} + r^{i\bar{j}} r_i r_{k\bar{j}}\\
&= -r^{i\bar{q}} r^{p\bar{j}} r_{p\bar{q}k} r_i r_{\bar{j}} + r^{i\bar{j}} r_{ik} r_{\bar{j}}\\
&= -r^{\bar{q}} r^p r_{p\bar{q}k} + r^i r_{ik} \qquad (4.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \log J}{\partial z_k} &= \frac{\partial \log \det H(r) + \log(-r + |\partial r|_r^2)}{\partial z_k}\\
&= \left(r^{i\bar{j}} - \frac{r^i r^{\bar{j}}}{-r + |\partial r|_r^2}\right) r_{i\bar{j}k} + \frac{r^i r_{ik}}{-r + |\partial r|_r^2}, \qquad (4.3)
\end{aligned}
$$

we have

$$R \log J(r)(z_0) = r^k \tilde{\Delta} r_k + \frac{r^i r^k}{|\partial r|_r^2} r_{ik}. \tag{4.4}$$

Thus,

$$\det H(\rho)(z^0)\, J(r)^{n/(n+1)}(z^0) = \det H(r)\Big(1 - \frac{2\mathrm{Re}\, r^k r^i r_{ik}}{(n+1)|\partial r|^2} + \tilde{E}(r)\Big), \tag{4.5}$$

where

$$\tilde{E}(r) =: \frac{|\partial r|^2}{n(n+1)}\Big[\tilde{\Delta} \log J(r) - \frac{n|\tilde{\nabla} \log J(r)|^2}{(n+1)} - 2n\mathrm{Re}\left(\frac{r^k \tilde{\Delta} r_k}{|\partial r|_r^2}\right)\Big]. \tag{4.6}$$

Proposition 4.1 *Let D be a smoothly bounded domain in the complex plane $\mathbb{C}$. Then D is (strictly) super-pseudoconvex if and only if D is (strictly) convex.*

Proof Let r be any smooth strictly subharmonic defining function on $D \subset \mathbb{C}$. By (4.5) and (4.6), we have $a^{1\overline{1}}[r] = 0$ and $\tilde{E}(r) = 0$ on ∂D. Therefore, D is strictly super-pseudoconvex if and only if

$$S_r(z) =: \det H(r)\Big(1 - \frac{2}{n+1}\mathrm{Re}\,\frac{r^k r^i r_{ik}}{|\partial r|_r^2}\Big) > 0 \tag{4.7}$$

on ∂D. For ant $z_0 \in \partial D$, by rotation, we may assume that $r_n(z_0) > 0$. Thus

$$S_r(z_0) = r_{1\overline{1}} - \mathrm{Re}\, r_{11}(z_0) \tag{4.8}$$

is positive for all $z_0 \in \partial D$ if and only if ∂D is strictly convex; and is non-negative for all $z_0 \in \partial D$ if and only if ∂D is convex, respectively. Therefore, the proof of the proposition is complete. $\square$

Next we estimate $\tilde{E}(r)$.

Proposition 4.2 *With the notation above, for $z \in \partial D$, we have*

$$\tilde{E}(r) \geq \frac{|\partial r|^2 a^{k\overline{\ell}}[r]}{n(n+1)}\Big[\tilde{\Delta} r_{k\overline{\ell}} - a^{i\overline{q}}[r] r^{p\overline{j}} r_{i\overline{j}k} r_{p\overline{q}\overline{\ell}} - (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\overline{\ell}}) - n\frac{r^i r_{ik} r^{\overline{j}} r_{\overline{j}\overline{\ell}}}{|\partial r|_r^4}\Big] \\ -\frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)} \tag{4.9}$$

and

$$\tilde{E}(r) \leq \frac{|\partial r|^2 a^{k\overline{\ell}}}{n(n+1)}\Big[\tilde{\Delta} r_{k\overline{\ell}} + a^{i\overline{q}}[r] r^p r^{\overline{j}} r_{i\overline{j}k} r_{p\overline{q}\overline{\ell}} + 2a^{i\overline{q}}[r]\,\frac{r_{ik} r_{\overline{q}\overline{\ell}}}{|\partial r|^2}\Big] - \frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)}. \tag{4.10}$$

Proof The following two identities will be used later.

$$(r^i)_{\overline{\ell}} = (r^{i\overline{q}}r_{\overline{q}})_{\overline{\ell}}=r_{\overline{q}}(r^{i\overline{q}})_{\overline{\ell}} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}}=-r^{i\overline{t}}r^{s\overline{q}}r_{s\overline{t}\overline{\ell}}r_{\overline{q}} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}}$$
$$= -r^{i\overline{t}}r^s r_{s\overline{t}\overline{\ell}} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}}$$

and

$$(r^{\overline{j}})_{\overline{\ell}} = (r^{p\overline{j}}r_p)_{\overline{\ell}} = -r^{\overline{q}}r^{i\overline{j}}r_{i\overline{q}\overline{\ell}} + \delta_{j\ell}.$$

By (4.3) and (4.2), for $z \in \partial D$, one has

$$\begin{aligned}
\frac{\partial^2 \log J(r)}{\partial z_k \partial \overline{z}_\ell} &= \left(r^{i\overline{j}} - \frac{r^i r^{\overline{j}}}{|\partial r|_r^2}\right) r_{i\overline{j}k\overline{\ell}} + r_{i\overline{j}k}\frac{\partial}{\partial \overline{z}_\ell}\left(r^{i\overline{j}} - \frac{r^i r^{\overline{j}}}{-r+|\partial r|_r^2}\right)\\
&\quad + \frac{\partial}{\partial \overline{z}_\ell}\frac{r^i r_{ik}}{(-r+|\partial r|_r^2)}\\
&= \tilde{\Delta} r_{k\overline{\ell}} - r_{i\overline{j}k}r^{i\overline{q}}r^{p\overline{j}}r_{p\overline{q}\overline{\ell}}\\
&\quad + \frac{1}{(|\partial r|_r^2)^2}(r_{i\overline{j}k}r^i r^{\overline{j}} - r^i r_{ik})\left(\frac{\partial(-r+|\partial r|_r^2)}{\partial \overline{z}_\ell}\right)\\
&\quad - \frac{r_{i\overline{j}k}}{|\partial r|_r^2}(r^i (r^{\overline{j}})_{\overline{\ell}} + r^{\overline{j}}(r^i)_{\overline{\ell}}) + \frac{1}{|\partial r|^2}(r^i r_{ik\overline{\ell}} + r_{ik}(r^i)_{\overline{\ell}})\\
&= \tilde{\Delta} r_{k\overline{\ell}} - r_{i\overline{j}k}r^{i\overline{q}}r^{p\overline{j}}r_{p\overline{q}\overline{\ell}}\\
&\quad + \frac{1}{(|\partial r|_r^2)^2}(r_{i\overline{j}k}r^i r^{\overline{j}} - r^i r_{ik})(-r^{\overline{q}}r^p r_{p\overline{q}\overline{\ell}} + r^{\overline{q}}r_{\overline{q}\overline{\ell}})\\
&\quad - \frac{r_{i\overline{j}k}}{|\partial r|_r^2}\left(r^{\overline{j}}(-r^{i\overline{t}}r^s r_{s\overline{t}\overline{\ell}} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}}) + r^i(-r^{\overline{q}}r^{p\overline{j}}r_{p\overline{q}\overline{\ell}} + \delta_{j\ell})\right)\\
&\quad + \frac{1}{|\partial r|^2}\left(r^i r_{ik\overline{\ell}} + r_{ik}(-r^{i\overline{t}}r^s r_{s\overline{t}\overline{\ell}} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}})\right)\\
&= \tilde{\Delta} r_{k\overline{\ell}} - r^{i\overline{q}}r^{p\overline{j}}r_{i\overline{j}k}r_{p\overline{q}\overline{\ell}} - \frac{1}{(|\partial r|_r^2)^2}(r_{i\overline{j}k}r^i r^{\overline{j}} - r^i r_{ik})(r^{\overline{q}}r^p r_{p\overline{q}\overline{\ell}} - r^{\overline{q}}r_{\overline{q}\overline{\ell}})\\
&\quad + \frac{1}{|\partial r|_r^2}(r^p r^{\overline{j}}r^{i\overline{q}} + r^i r^{\overline{q}}r^{p\overline{j}})r_{p\overline{q}\overline{\ell}}r_{i\overline{j}k} - \frac{1}{|\partial r|_r^2}r^{\overline{j}}r^{i\overline{q}}r_{\overline{q}\overline{\ell}}r_{i\overline{j}k} - \frac{r_{i\overline{\ell}k}}{|\partial r|_r^2}r^i\\
&\quad + \frac{1}{|\partial r|^2}\left(r^i r_{ik\overline{\ell}} - r^{i\overline{t}}r^s r_{s\overline{t}\overline{\ell}}r_{ik} + r^{i\overline{q}}r_{\overline{q}\overline{\ell}}r_{ik}\right)\\
&= \tilde{\Delta} r_{k\overline{\ell}} - r^{i\overline{q}}r^{p\overline{j}}r_{i\overline{j}k}r_{p\overline{q}\overline{\ell}} - \frac{r^i r^{\overline{j}}r^p r^{\overline{q}}}{|\partial r|_r^4}r_{i\overline{j}k}r_{p\overline{q}\overline{\ell}} + \frac{1}{|\partial r|_r^4}(r^i r^{\overline{j}}r_{i\overline{j}k}r^{\overline{q}}r_{\overline{q}\overline{\ell}}\\
&\quad + r^p r^{\overline{q}}r_{p\overline{q}\overline{\ell}}r^i r_{ik})\\
&\quad + \frac{1}{|\partial r|_r^2}(r^p r^{\overline{j}}r^{i\overline{q}} + r^i r^{\overline{q}}r^{p\overline{j}})r_{p\overline{q}\overline{\ell}}r_{i\overline{j}k}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{|\partial r_r^2}(r^i r^{p\bar{j}} r_{pk} r_{i\bar{j}\bar{\ell}} + r^{\bar{j}} r^{i\bar{q}} r_{\bar{q}\bar{\ell}} r_{i\bar{j}k}) + \frac{1}{|\partial r|_r^2}\left(r^{i\bar{q}} - \frac{r^i r^{\bar{q}}}{|\partial r|_r^2}\right) r_{\bar{q}\bar{\ell}} r_{ik} \\
&= \tilde{\Delta} r_{k\bar{\ell}} - \left(r^{i\bar{q}} - \frac{r^i r^{\bar{q}}}{|\partial r|_r^2}\right)\left(r^{p\bar{j}} - \frac{r^p r^{\bar{j}}}{|\partial r|_r^2}\right) r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} \\
&\quad -\frac{1}{|\partial r|_r^2}\left(r^i\left(r^{p\bar{j}} - \frac{r^p r^{\bar{j}}}{|\partial r|_r^2}\right) r_{pk} r_{i\bar{j}\bar{\ell}} + r^{\bar{j}}\left(r^{i\bar{q}} - \frac{r^i r^{\bar{q}}}{|\partial r|_r^2}\right) r_{\bar{q}\bar{\ell}} r_{i\bar{j}k}\right) \\
&\quad +\frac{1}{|\partial r|_r^2}\left(r^{i\bar{q}} - \frac{r^i r^{\bar{q}}}{|\partial r|_r^2}\right) r_{\bar{q}\bar{\ell}} r_{ik}.
\end{aligned}
$$

Then for $z \in \partial D$, we have

$$
\begin{aligned}
\tilde{\Delta} \log J(r)(z) &\geq a^{k\bar{\ell}}[r] \tilde{\Delta} r_{k\bar{\ell}} - a^{k\bar{\ell}}[r] a^{i\bar{q}}[r] a^{p\bar{j}}[r] r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} \\
&\quad - a^{k\bar{\ell}}[r] \frac{a^{i\bar{q}}[r]}{|\partial r|_r^2}\left(r^{\bar{j}} r_{i\bar{j}k} r^p r_{p\bar{q}\bar{\ell}} + r_{ki} r_{\bar{q}\bar{\ell}}\right) + \frac{1}{|\partial r|_r^2} a^{k\bar{\ell}}[r] a^{i\bar{q}}[r] r_{\bar{q}\bar{\ell}} r_{ik} \\
&= a^{k\bar{\ell}} \tilde{\Delta} r_{k\bar{\ell}} - a^{k\bar{\ell}}[r] a^{i\bar{q}}[r]\, r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}}
\end{aligned}
$$

and

$$
\tilde{\Delta} \log J(r)(z) \leq a^{k\bar{\ell}} \tilde{\Delta} r_{k\bar{\ell}} + 2 a^{k\bar{\ell}}[r] a^{i\bar{q}}[r] \frac{r_{ik} r_{\bar{q}\bar{\ell}}}{|\partial r|^2} + a^{k\bar{\ell}}[r] a^{i\bar{q}}[r] r^p r^{\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}}.
$$

Moreover,

$$
\begin{aligned}
|\tilde{\nabla} \log J(r)|^2 &= a^{k\bar{\ell}}[r] \left(\tilde{\Delta} r_k + \frac{r^i r_{ik}}{|\partial r|_r^2}\right)\left(\tilde{\Delta} r_{\bar{\ell}} + \frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right) \\
&= a^{k\bar{\ell}}[r] \left[(\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}}) + (\tilde{\Delta} r_k)\left(\frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right) + \frac{r^i r_{ik}}{|\partial r|_r^2} \tilde{\Delta} r_{\bar{\ell}} + \frac{r^i r_{ik}}{|\partial r|_r^2} \frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right] \\
&\leq a^{k\bar{\ell}}[r] \left[\frac{n+1}{n} (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}}) + (n+1) \frac{r^i r_{ik}}{|\partial r|_r^2} \frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{\Delta} \log J(r) - \frac{n}{n+1} |\tilde{\nabla} \log J|^2 &\geq a^{k\bar{\ell}}[r]\left(\tilde{\Delta} r_{k\bar{\ell}} - a^{i\bar{q}}[r] r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}}\right) \\
&\quad - a^{k\bar{\ell}}[r]\left((\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}}) + n \frac{r^i r_{ik}}{|\partial r|_r^2} \frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right).
\end{aligned}
$$

Therefore,

$$
\tilde{E}(r) \geq \frac{|\partial r|^2 a^{k\bar{\ell}}[r]}{n(n+1)}\left(\tilde{\Delta} r_{k\bar{\ell}} - a^{i\bar{q}}[r] r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} - (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}}) - n \frac{r^i r_{ik}}{|\partial r|_r^2} \frac{r^{\bar{j}} r_{\overline{j\ell}}}{|\partial r|_r^2}\right)
$$

$$-\frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)}$$

and

$$\tilde{E}(r) \le \frac{|\partial r|^2 a^{k\bar{\ell}}[r]}{n(n+1)}\Big[\tilde{\Delta} r_{k\bar{\ell}} + a^{i\bar{q}} r^p r^{\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} + 2a^{i\bar{q}}[r]\frac{r_{ik} r_{\bar{q}\bar{\ell}}}{|\partial r|^2}\Big] - \frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)}$$

Therefore, the proof of the proposition is complete. ◻

Corollary 4.3 *Let D be smoothly bounded convex domain in $\mathbb{C}^n$. If there is a strictly plurisubharmonic defining function $r \in C^4(\overline{D})$ such that*

$$\begin{aligned} &\frac{n-1}{n+1} + \frac{|\partial r|^2 a^{k\bar{\ell}}[r]}{n(n+1)}\Big(\tilde{\Delta} r_{k\bar{\ell}} - a^{i\bar{q}}[r] r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} - (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}})\Big) \\ &\quad - \frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)} > 0 \ on \ \ \partial D, \end{aligned} \tag{4.11}$$

then D is strictly super-pseudoconvex.

Proof If ∂D is convex then for any strictly plurisubharmonic defining function $r \in C^4(\overline{D})$, we have

$$\frac{2}{n+1} - \frac{2}{n+1}\mathrm{Re}\frac{r^k r^i r_{ik}}{|\partial r|^2} - \frac{a^{k\bar{\ell}}[r] r^i r_{ik} r^{\bar{j}} r_{\overline{j\ell}}}{(n+1)|\partial r|_r^2} \ge 0 \quad \text{on } \partial D. \tag{4.12}$$

Since

$$\begin{aligned} \tilde{E}(r) + \frac{1}{n+1} a^{k\bar{\ell}}[r] r^i r_{ik} r^{\bar{j}} r_{\overline{j\ell}} &= \frac{|\partial r|^2 a^{k\bar{\ell}}[r]}{n(n+1)} \\ &\quad \times \Big(\tilde{\Delta} r_{k\bar{\ell}} - a^{i\bar{q}}[r] r^{p\bar{j}} r_{i\bar{j}k} r_{p\bar{q}\bar{\ell}} - (\tilde{\Delta} r_k)(\tilde{\Delta} r_{\bar{\ell}})\Big) \quad - \frac{2\mathrm{Re}\, r^k \tilde{\Delta} r_k}{(n+1)} \end{aligned}$$

and $1 - \frac{2}{n+1} = \frac{n-1}{n+1}$, by (4.5), (4.11) and (4.12), we have $\det H(\rho) > 0$ on ∂D. This implies ρ is strictly plurisubharmonic on $\overline{D}$ by Lemma 2 in [20]. This proves Parts (i) and (ii) in Theorem 1.4. ◻

5 Examples

In this section, we will provide two examples in $\mathbb{C}^2$ which give the proof of Part (iii) of Theorem 1.4.

For $\delta = 4^{-12}$, we let

$$g(t) =: g_\delta(t) =: \begin{cases} e^{-\frac{\delta}{\delta - t}}, & \text{if } t < \delta, \\ 0, & \text{if } t \ge \delta. \end{cases} \tag{5.1}$$

Let

$$r(z) = -2\mathrm{Re}\, z_2 + |z|^2 - 8|z_1|^4 g(|z_1|^2), \quad z = (z_1, z_2) \in \mathbb{C}^2. \tag{5.2}$$

Example 1 Let $D = \{z \in \mathbb{C}^2 : r(z) < 0\}$. Then

(i) D is strictly convex.
(ii) If ρ_D the solution of Fefferman equation (2), then ρ_D is not plurisubharmonic in D.

Proof Since

$$\begin{aligned}
\frac{\partial |z_1|^4 g(|z_1|^2)}{\partial x_1} &= 4|z_1|^2 x_1 g(|z_1|^2) + |z_1|^4 g'(|z_1|^2) 2x_1,\\
\frac{\partial |z_1|^4 g(|z_1|^2)}{\partial y_1} &= 4|z_1|^2 y_1 g(|z_1|^2) + |z_1|^4 g'(|z_1|^2) 2y_1,\\
\frac{\partial^2\, |z_1|^4 g(|z_1|^2)}{\partial x_1^2} &= 16|z_1|^2 x_1^2 g'(|z_1|^2) + 2|z_1|^4 g'(|z_1|^2) + 4(|z_1|^2 + 2x_1^2) g(|z_1|^2)\\
&\quad +4|z_1|^4 g''(|z_1|^2) x_1^2,\\
\frac{\partial |z_1|^4 g(|z_1|)}{\partial y_1^2} &= 16|z_1|^2 y_1^2 g'(|z_1|^2) + 2|z_1|^4 g'(|z_1|^2) + 4(|z_1|^2 + 2y_1^2) g(|z_1|^2)\\
&\quad +4|z_1|^4 g''(|z_1|^2) y_1^2,\\
\frac{\partial^2 (|z_1|^4 g(|z_1|^2)}{\partial x_1 \partial y_1} &= \frac{\partial (4|z_1|^2 x_1 g(|z_1|^2) + |z_1|^4 g'(|z_1|^2) 2x_1)}{\partial y_1}\\
&= 8x_1 y_1 g(|z_1|^2) + 16|z_1|^2 x_1 y_1 g'(|z_1|^2) + 4|z_1|^4 x_1 y_1 g''(|z_1|^2);
\end{aligned}$$

and since

$$\begin{aligned}
20t^2|g'(t)| + 12tg(t) + 4t^3|g''(t)| &= 4tg(t)\left[3 + 5\frac{t\delta}{(\delta - t)^2} + \frac{t^2(\delta^2 + 2\delta(\delta - t))}{(\delta - t)^4}\right]\\
&\le 4tg(t)\left[\frac{11\delta^4}{(\delta - t)^4}\right]\\
&\le 4^7\delta\\
&\le 4^{-5},
\end{aligned}$$

we have

$$\begin{aligned}
&18|z_1|^4|g'(|z_1|^2)| + 12|z_1|^2 g(|z_1|^2) + 4|z_1|^6|g''(|z_1|^2)| \le 1/4,\\
&\left|\frac{\partial(|z_1|^4 g(|z_1|^2)}{\partial x_1^2}\right| < 1/4,\\
&\left|\frac{\partial(|z_1|^4 g(|z_1|^2)}{\partial y_1^2}\right| < 1/4
\end{aligned}$$

and

$$\left|\frac{\partial(|z_1|^4 g(|z_1|^2)}{\partial x_1 \partial y_1}\right| < 1/2.$$

Therefore, $D^2 r(z) = 2I_n + D^2(|z_1|^4 g(|z_1|^2))$ is positive definite in $\mathbb{R}^4$. Therefore, D is strictly convex. Moreover, $H(r)(0) = I_2$. We claim that

$$\det H(\rho_D)(0) < 0.$$

Since, at $z = 0$, we have

$$\frac{\partial r}{\partial z_2} = -1, \quad r_{kj}(0) = r_{i\bar{j}k}(0) = 0, \quad 1 \le i, j, k \le 2$$

By (4.3). This implies $\frac{\partial \log J(r)}{\partial z_j}(0) = 0$ for all $1 \le j \le 2$. By (4.6) and (4.10), we have

$$r_{1\bar{1}1\bar{1}}(0) = -32e^{-1}, \quad \tilde{E}(r)(0) = \frac{|\partial r|^2}{6} r_{1\bar{1}1\bar{1}}(0) = -\frac{32}{6}e^{-1}$$

Thus,

$$\det H(\rho_D) J(r)^{2/3} = 1 - \frac{2}{3} - \frac{32}{6e} < 0.$$

This completes the proof of the statement in the example. □

Example 2 For $n \ge 2, \alpha = 21/20$ and $0 < C \le (9 - 8\alpha)(1 + \alpha)/256$, we let $r(z) = |z|^2 + 2\mathrm{Re}\, z_n + \alpha \mathrm{Re} \sum_{j=1}^n z_j^2 + C \sum_{j=1}^n |z_j|^4$ and let

$$D = \{z \in \mathbb{C}^n : r(z) < 0\}$$

Then D is super-pseudoconvex, but D is not convex.

Proof At $z = (0, 0, \dots, 0) \in \partial D$, we have that $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial y_n}$ are tangent vectors to ∂D for $1 \le j \le n - 1$. Notice that

$$\frac{\partial^2 r}{\partial y_n^2} = 2 - 2\alpha = -2(\alpha - 1) < 0,$$

one can easily see that ∂D not convex at $z = 0$. Thus, ∂D is not convex. However,

$$H(r) = I_n + 4C\mathrm{Diag}(|z_1|^2, \dots, |z_n|^2),$$

where $\mathrm{Diag}(|z_1|^2, \ldots, |z_n|^2)$ is a diagonal matrix with diagonal entries $|z_1|^2, \ldots, |z_n|^2$, respectively. Then

$$\frac{\partial^2 r}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_\ell}(z) = 4C\delta_{ij}\delta_{k\ell}\delta_{ik}, \quad \frac{\partial^3 r}{\partial z_k \partial \bar{z}_\ell \partial z_j} = 4C\delta_{k\ell}\delta_{kj}\bar{z}_j,$$
$$\frac{\partial^2 r}{\partial z_i \partial z_j} = (\alpha + 2C\bar{z}_j^2)\delta_{ij}.$$

For each i

$$r^i = \frac{r_{\bar{i}}}{1+4C|z_i|^2}, \quad |\partial r|_r^2 = r^i r_i = \sum_{i=1}^n \frac{|r_i|^2}{1+4C|z_i|^2}$$

and, on ∂D, we have

$$\tilde{\Delta} = \sum_{i,j=1}^n \left(\frac{\delta_{ij}}{1+4C|z_j|^2} - \frac{r_{\bar{i}} r_j}{(1+4C|z_i|^2)(1+4C|z_j|^2)|\partial r|_r^2} \right) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Notice that if $z \in D$, then

$$2x_n + (1+\alpha)\sum_{j=1}^n x_j^2 + (1-\alpha)\sum_{j=1}^n y_j^2 + C\sum (x_j^2 + y_j^2)^2 < 0.$$

This implies that

$$2x_n + (1+\alpha)x_n^2 < 0 \iff -\frac{2}{1+\alpha} < x_n < 0. \tag{5.3}$$

Thus

$$2x_n + (1+\alpha)x_n^2 > \frac{-1}{1+\alpha} \quad \text{and} \quad C|z_k|^4 - (\alpha - 1)|z_k|^2 < \frac{1}{1+\alpha}. \tag{5.4}$$

We claim that

$$4C|z_k|^2 \le 1/8 \quad \text{if} \quad 0 < C \le \frac{(9-8\alpha)(1+\alpha)}{256}, \quad 1 < \alpha < 9/8. \tag{5.5}$$

Otherwise, $4C|z_k|^2 \ge 1/8$. Therefore, $C|z_k|^4 - (\alpha - 1)|z_k|^2 < \frac{1}{1+\alpha}$ implies

$$|z_k|^2 < \frac{8}{(1+\alpha)(9-8\alpha)}.$$

This is a contradiction with $4C|z_k| \ge 1/8$. Therefore, the claim is true. Notice

$$a^{k\bar{\ell}}[r] r_{\bar{\ell}} = 0, \quad \text{for all } 1 \le k \le n,$$

we have

$$
\begin{aligned}
\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)\left(r^ir_{ik}r^{\bar j}r_{\overline{j\ell}}\right)&=\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)r^kr^{\bar\ell}(\alpha+2C\bar z_k^2)(\alpha+2Cz_\ell^2)\\
&=\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)r_kr_{\bar\ell}\left(\alpha+2C\frac{\bar z_k^2-2\alpha|z_k|^2}{1+4C|z_k|^2}\right)\\
&\quad\times\left(\alpha+2C\frac{z_\ell^2-2\alpha|z_\ell|^2}{1+4C|z_\ell|^2}\right)\\
&=\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)r_kr_{\bar\ell}\alpha^2\\
&\quad+4C\alpha\mathrm{Re}\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)r_kr_{\bar\ell}\frac{\bar z_k^2-2\alpha|z_k|^2}{1+4C|z_k|^2})\\
&\quad+4C^2\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)r_kr_{\bar\ell}\frac{(\bar z_k^2-2\alpha|z_k|^2)(z_\ell^2-2\alpha|z_\ell|^2)}{(1+4C|z_k|^2)(1+4C|z_\ell|^2)}\\
&\le\frac{4C^2(2\alpha+1)^2|z_k|^4}{(1+4C|z_k|^2)^2}r^{k\bar k}|r_k|^2\\
&\le\frac{(2\alpha+1)^2}{256}|\partial r|^2,
\end{aligned}
$$

$$
\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|_r^2}\right)\tilde\Delta r_{k\bar\ell}=4C\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|_r^2}\right)\tilde\Delta|z_k|^2=4C\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|_r^2}\right)^2,
$$

$$
\tilde\Delta r_k=4C\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|^2}\right)\bar z_k
$$

and

$$
r_{\bar k}=(1+2C|z_k|^2)z_k+2\alpha\bar z_k.
$$

Thus by (5.3)

$$
\begin{aligned}
\mathrm{Re}\, r^k\tilde\Delta r_k&=4C\mathrm{Re}\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|^2}\right)r^k\bar z_k\\
&\le 4C\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|^2}\right)r^{k\bar k}(1+2\alpha+2C|z_k|^2)|z_k|^2\\
&=\frac{4C|z_k|^2(1+2\alpha+2C|z_k|^2)}{(1+4C|z_k|^2)^2}\le\frac{2\alpha+1}{8},
\end{aligned}
$$

$$
\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)\tilde\Delta r_k\tilde\Delta r_{\bar\ell}=16C^2\left(r^{k\bar\ell}-\frac{r^kr^{\bar\ell}}{|\partial r|^2}\right)\bar z_kz_\ell\left(r^{k\bar k}-\frac{r^kr^{\bar k}}{|\partial r|^2}\right)\left(r^{\ell\bar\ell}-\frac{r^\ell r^{\bar\ell}}{|\partial r|^2}\right)
$$

$$\le 16C^2 r^{k\bar\ell}\bar z_k z_\ell\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)\left(r^{\ell\bar\ell}-\frac{r^\ell r^{\bar\ell}}{|\partial r|^2}\right)$$
$$\le 4C\frac{4C|z_k|^2}{1+4C|z_k|^2}\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)^2$$

and

$$\begin{aligned}\left(r^{k\bar\ell}-\frac{r^k r^{\bar\ell}}{|\partial r|^2}\right)\left(r^{i\bar q}-\frac{r^i r^{\bar q}}{|\partial r|^2}\right)r^{p\bar j}r_{i\bar j k}r_{p\bar q\bar\ell}&=16C^2\left(r^{k\bar\ell}-\frac{r^k r^{\bar\ell}}{|\partial r|^2}\right)^2 r^{\ell\bar k}\bar z_k\delta_{ik}\delta_{jk}z_\ell\delta_{p\ell}\delta_{q\ell}\\&=16C^2|z_k|^2\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)^2 r^{k\bar k}\\&=4C\frac{4C|z_k|^2}{(1+4C|z_k|^2)}\left(r^{k\bar k}-\frac{r^{kk}|r_k|^2}{|\partial r|^2}\right)^2.\end{aligned}$$

Therefore, since (5.1), we have

$$\begin{aligned}&\left(r^{k\bar\ell}-\frac{r^k r^{\bar\ell}}{|\partial r|_r^2}\right)\tilde\Delta r_{k\bar\ell}-\left(r^{k\bar\ell}-\frac{r^k r^{\bar\ell}}{|\partial r|^2}\right)\left(r^{i\bar q}-\frac{r^i r^{\bar q}}{|\partial r|^2}\right)r^{p\bar j}r_{i\bar j k}r_{p\bar q\bar\ell}\\&\quad-4C\frac{4C|z_k|^2}{1+4C|z_k|^2}\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)^2\\&=4C\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|_r^2}\right)^2-4C\frac{4C|z_k|^2}{(1+4C|z_k|^2)}\left(r^{k\bar k}-\frac{r^{kk}|r_k|^2}{|\partial r|^2}\right)^2\\&\quad-4C\frac{4C|z_k|^2}{1+4C|z_k|^2}\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)^2\\&=4C(1-2\frac{4C|z_k|^2}{1+4C|z_k|^2})\left(r^{k\bar k}-\frac{r^k r^{\bar k}}{|\partial r|^2}\right)^2\\&\ge 0.\end{aligned}$$

Therefore,

$$\tilde E(r)\ge-\frac{2\mathrm{Re}\,r^k\tilde\Delta r_k}{n+1}-a^{k\bar\ell}[r]\frac{r^i r_{ik}r^{\bar j}r_{\overline{j\ell}}}{(n+1)|\partial r|^2}\ge-\frac{(1+2\alpha)}{4(n+1)}-\frac{(2\alpha+1)^2}{256(n+1)}$$

and

$$\begin{aligned}
1-\frac{2}{n+1}\mathrm{Re}\,\frac{r^i r^k r_{ik}}{|\partial r|^2}+\tilde{E}(r) &\geq 1-\frac{2}{n+1}\mathrm{Re}\,\frac{r^i r^i(\alpha+2C\bar{z}_i^2)}{|\partial r|^2}-\frac{(1+2\alpha)}{4(n+1)}\\
&\quad -\frac{(2\alpha+1)^2}{256(n+1)}\\
&= 1-\frac{2}{n+1}\mathrm{Re}\,\frac{r^{i\bar{i}}r_i^2 r^{i\bar{i}}(\alpha+2C\bar{z}_i^2)}{|\partial r|^2}-\frac{(1+2\alpha)}{4(n+1)}\\
&\quad -\frac{(2\alpha+1)^2}{256(n+1)}\\
&\geq 1-\frac{2\alpha}{n+1}-\frac{(1+2\alpha)}{4(n+1)}-\frac{(2\alpha+1)^2}{256(n+1)}\\
&> 1-\frac{10\alpha+1}{4(n+1)}-\frac{10}{256(n+1)}\\
&\geq 1-\frac{23}{24}-\frac{1}{25}\\
&> 0
\end{aligned}$$

if $n \geq 2$ and $\alpha \leq 21/20$. Therefore, by (1.13) in Definition 1.1 and (4.5) and (4.6), D is strictly super-pseudoconvex and the proof is complete. □

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